Symbolic Computation Study of a Generalized Variable-Coefficient Two-Dimensional Korteweg-de Vries Model with Various External-Force Terms from Shallow Water Waves, Plasma Physics, and Fluid Dynamics

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The variable-coefficient two-dimensional Korteweg-de Vries (KdV) model is of considerable significance in describing many physical situations such as in canonical and cylindrical cases, and in the propagation of surface waves in large channels of varying width and depth with nonvanishing vorticity. Under investigation hereby is a generalized variable-coefficient two-dimensional KdV model with various external-force terms. With the extended bilinear method, this model is transformed into a variable-coefficient bilinear form, and then a Bäcklund transformation is constructed in bilinear form. Via symbolic computation, the associated inverse scattering scheme is simultaneously derived on the basis of the aforementioned bilinear Bäcklund transformation. Certain constraints on coefficient functions are also analyzed and finally some possible cases of the external-force terms are discussed.

Key words: Generalized Variable-Coefficient Two-Dimensional Korteweg-de Vries Model; Symbolic Computation; Bäcklund Transformation; Variable-Coefficient Bilinear Form; Inverse Scattering Scheme.

1. Motivations for a Generalized Variable-Coefficient Two-Dimensional Korteweg-de Vries Model with Various External-Force Terms from Shallow Water Waves, Plasma Physics, and Fluid Dynamics

Among the most important models of nonlinear evolution equations (NLEEs), constant-coefficient Korteweg-de Vries (KdV) and KdV-type equations are encountered in many apparently unrelated phenomena in different physical areas such as shallow water waves, plasmas, fluids and lattice vibrations of a crystal at low temperatures [1]. In all of these applications, the physical situations described via the KdV (or KdV-type) models tend to be highly idealized, owing to the assumption of constant coefficients. Considering the inhomogeneities of media, nonuniformities of boundaries and external forces, the variable-coefficient models are much more powerful and realistic than their constant-coefficient counterparts in describing various situations, e. g., in the coastal waters of oceans, space and laboratory plasmas, superconductors and optical-fiber communications [2 – 6].

In the past decades, it has been shown that various physical phenomena in nature, actual physics and engineering can be described by the variable-coefficient KdV model with perturbed, dissipative and external-force terms as [7]

$$\begin{aligned} &v_t + \mu_1(t) v v_x + \mu_2(t) v_{xxx} \\ &+ \mu_3(t) v_x + \mu_4(t) v = \mu_5(t), \end{aligned} \tag{1}$$

where $v(x,t)$ is a function of the variables x and t, $\mu_1(t) \neq 0$, $\mu_2(t) \neq 0$, $\mu_3(t)$, $\mu_4(t)$ and $\mu_5(t)$ represent the coefficients of the nonlinear, dispersive, dissipative, perturbed and external-force terms, respectively, all of which are real functions. In recent studies, of physical and mechanical interests, many important examples of (1), among others, can be listed [8]:

- $\Phi_t + h(t)\Phi\Phi_x + g(t)\Phi_{xxx} = 0,$ (2)

- $\Psi_t + l(t)\Psi\Psi_x + m(t)\Psi_{xxx} + n(t)\Psi = 0,$ (3)
- $\Omega_t + p(t)\Omega\Omega_x + q(t)\Omega_{xxx} + r(t)\Omega_x + s(t)\Omega = 0.$ (4)

Note that, if we set

$$\mu_1(t) = 6, \quad \mu_2(t) = 1, \quad \mu_4(t) = \frac{1}{2t}, \\ \mu_3(t) = \mu_5(t) = 0,$$

then one important case of (1), the celebrated cylindrical KdV equation [9, 10], is obtained as

$$v_t + 6v\,v_x + v_{xxx} + \frac{1}{2t}v = 0. \quad (5)$$

On the other hand, it is of great interest to extend the KdV model to higher dimension, e. g., to the two-dimensional KdV model (also called the Kadomtsev-Petviashvili model), which has many physical applications from water waves to field theories, plasma physics and fluid dynamics [10, 11]. What's more, the variable-coefficient generalizations of the two-dimensional KdV model [6, 12 – 15] are considered to be more realistic in modeling physical situations. Arising from various branches of physics, some important variable-coefficient generalizations of the two-dimensional KdV models can be seen below [12, 14, 15]:

- $[V_t + m_1(t)V + m_2(t)V\,V_x + m_3(t)V_{xxx}]_x + m_4(t)V_{yy} + m_5(t) = 0,$ (6)
- $[U_t + 6U\,U_x + U_{xxx}]_x + \kappa(t)U_x + \nu(t)U_{yy} = 0,$ (7)
- $[W_t + W\,W_x + W_{xxx}]_x + n_1(y,t)W_x + n_2(y,t)W_y + n_3(y,t)W_{yy} + n_4(y,t)W_{xy} + n_5(y,t)W_{xx} = 0.$ (8)

It is clear that, if we set

$$m_1(t) = \frac{1}{2t}, \quad m_2(t) = 6, \quad m_3(t) = 1, \quad m_5(t) = 0, \\ m_4(t) = \frac{3\sigma^2}{t^2} \text{ with } \sigma^2 = \pm 1$$

in (6), the celebrated cylindrical Kadomtsev-Petviashvili equation [10, 16] is obtained as

$$[V_t + 6V\,V_x + V_{xxx}]_x + \frac{1}{2t}V_x + \frac{3\sigma^2}{t^2}V_{yy} = 0. \quad (9)$$

Generally speaking, the considerable significance of the variable-coefficient KdV and two-dimensional KdV models mainly lies in: (A) the models arise in many physical applications including shallow water waves, plasma physics, and fluid dynamics and are able to describe various situations more powerful than their constant-coefficient counterparts; (B) the investigations on the approaches and techniques to manage the models are various and of great interest; (C) the mathematical interest is whether the models are integrable or not.

Nowadays, there has been a growing interest in studying variable-coefficient NLEEs [4, 15, 17 – 19], which provide a large family of powerful models for describing the real-world situations in many fields of physical and engineering sciences. Considering *the inhomogeneities of media, nonuniformities of boundaries* and *external forces*, we hereby investigate *a generalized variable-coefficient two-dimensional KdV model with external-force term* [12]:

$$[u_t + 6u\,u_x + u_{xxx}]_x + a(t)u_x \\ + b(t)u_{yy} + F_{xx}(x,y,t) = 0, \quad (10)$$

where $u(x,y,t)$ is a function of the variables x, y and t, $a(t)$ and $b(t)$ depend only on the variable t, while the external-force term, $F_{xx}(x,y,t)$, depends on the variables x, y and t.

It can be seen that the introduction of variable-coefficient functions into the NLEEs usually gives rise to many difficulties in the investigations [2 – 7, 12, 15] owing to the involvement of a great amount of integral and differential calculations which are manually unmanageable. With the development of computer science and technology, symbolic computation as a new branch of artificial intelligence drastically increases the ability of a computer to deal with the complicated and tedious calculations of the coefficient functions in the NLEEs under investigation. In soliton theory, the computerized symbolic computation system like MATHEMATICA or MAPLE [17 – 22] has been playing a more and more important role in analytically investigating NLEEs, such as, transforming a wide class of variable-coefficient models to the standard ones [17, 18], constructing exact analytical solutions [17, 19, 22] (including the solitonic solutions, periodic solutions, rational solutions and so on), and testing the integrability of NLEEs [17, 20, 21].

By means of the symbolic computation system MATHEMATICA, in the present paper, we will extend and perform the bilinear method which has been shown to be powerful in application of constant-coefficient NLEEs to a variable-coefficient equation, i. e., (10).

2. Symbolic Computation Study of (10) with the Extended Bilinear Method

Hirota's bilinear method has been extensively studied and widely used [2, 4, 23, 24]. The fundamental idea behind the method is using a suitable dependent variable transformation (DVT) to put the original NLEE in a form where the new dependent variable appears bilinear. Once the bilinear form of the equation is found, one may employ the perturbation technique to construct its solution step by step. If soliton solutions exist this expansion will always truncate and then finite series will lead to an exact solution. Many perfect software packages available for the bilinear method have already been proposed see [21] for details.

2.1. Variable-Coefficient Bilinear Form

Substituting the DVT: $u(x,y,t) = 2[\mathrm{Log} f(x,y,t)]_{xx}$ into (10), integrating twice with respect to x, and taking the integration constants as zero, we get *the variable-coefficient bilinear form* of (10) as

$$\begin{aligned}&\big[D_xD_t + D_x^4 + b(t)D_y^2 + a(t)\partial_x \\ &\quad + 2F(x,y,t)\big](f\cdot f) = 0,\end{aligned} \tag{11}$$

where the binary operators D_x, D_t and D_y [23, 24] are defined by

$$D_x^m D_y^k D_t^n(\theta\cdot\rho) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^k \cdot \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \theta(x,y,t)\rho(x',y',t')\Big|_{x'=x,y'=y,t'=t}.$$

2.2. Construction of the Bilinear Bäcklund Transformation Based on (11)

With the help of symbolic computation, now we begin to construct a bilinear Bäcklund transformation between f and g, both of which are assumed to be two different solutions of (11) [simultaneously be solutions of (10) through the aforementioned DVT], by considering [2, 4, 24]

$$\begin{aligned}0 = P \equiv\ & f^2[D_xD_t + D_x^4 + b(t)D_y^2 + a(t)\partial_x \\ & + 2F(x,y,t)](g\cdot g) - g^2[D_xD_t + D_x^4 \\ & + b(t)D_y^2 + a(t)\partial_x + 2F(x,y,t)](f\cdot f) \\ =\ & 2D_x[D_t(g\cdot f)\cdot(gf)] + 2D_x[D_x^3(g\cdot f)\cdot(gf)] \\ & + 6D_x[D_x^2(g\cdot f)\cdot D_x(f\cdot g)] \\ & + 2b(t)D_y[D_y(g\cdot f)\cdot(gf)] \\ & + D_x\{[a(t)xD_x(g\cdot f)]\cdot(gf) \\ & - [a(t)xgf]\cdot D_x(f\cdot g)\}.\end{aligned} \tag{12}$$

Taking into account: $D_y[D_x^2(g\cdot f)\cdot(gf)] = D_y\{[D_xD_y(g\cdot f)]\cdot(gf)\} + D_x\{[D_y(g\cdot f)]\cdot[D_x(g\cdot f)]\}$, we can add $+6B(t)D_y[D_x^2(g\cdot f)\cdot(gf)] - 6B(t)\cdot D_y[D_x^2(g\cdot f)\cdot(gf)]$ with $B(t)\neq 0$ as an arbitrary function of t to (12) and obtain

$$\begin{aligned}P =\ & D_x\{[2D_t + 2D_x^3 + 6B(t)D_xD_y + a(t)xD_x](g\cdot f)\cdot(gf)\} \\ & + D_x\{[6D_x^2 - 6B(t)D_y - a(t)x](g\cdot f)\cdot[D_x(f\cdot g)]\} \\ & + D_y\{[-6B(t)D_x^2(g\cdot f) + 2b(t)D_y(g\cdot f) \\ & \quad + k(t)xgf]\cdot(gf)\},\end{aligned}$$

where the arbitrary function $k(t)$ has been introduced.

To this stage, with three decoupling functions $\chi(t)$, $\lambda(t)$ and $\mu(y,t)$, all of which are arbitrary, equation splitting indicates that

$$[-6B(t)D_x^2 + 2b(t)D_y + k(t)x](g\cdot f) = \chi(t)gf, \tag{13}$$

$$[6D_x^2 - 6B(t)D_y - a(t)x](g\cdot f) = \lambda(t)gf, \tag{14}$$

$$\begin{aligned}&[2D_t + 2D_x^3 + 6B(t)D_xD_y + a(t)xD_x + \lambda(t)D_x](g\cdot f) \\ &= \mu(y,t)gf.\end{aligned} \tag{15}$$

It is obvious that once g and f satisfy (13), (14) and (15), then $P = 0$. Accordingly, it is reasonable to treat the set of (13), (14) and (15) as *a formally generalized Bäcklund transformation* for (11) [and/or (10)]. In other words, supposing that f is a solution of (11), if g with f satisfies (13), (14) and (15), then we can conclude that g is a solution of (11) as well.

Having in mind *the arbitrariness* of functions $B(t)$, $k(t)$ and $\chi(t)$, and choosing $k(t) = a(t)B(t)$ and $\chi(t) = -B(t)\lambda(t)$ with $3B^2(t) = b(t)$, we can rewrite (13) as

$$\begin{aligned}-B(t)[6D_x^2 - 6B(t)D_y - a(t)x](g\cdot f) &= \chi(t)gf \\ &= -B(t)\lambda(t)gf,\end{aligned}$$

which, obviously, is equivalent to (14). Therefore, we can take the set of (14) and (15), in a more compact and clear way, as a bilinear Bäcklund transformation for (11) [and/or (10)].

In order to illustrate that (14) and (15) do construct a Bäcklund transformation, we need to investigate their compatibility condition. This compatibility, to some extent, is related to the inverse scattering scheme, which will be conducted in the next section. As a result, we will get certain constraints on the variable coefficients and the external-force term.

3. Derivation and Investigation of the Inverse Scattering Scheme for (10)

In [25], with the help of a commutativity condition, the authors have constructed Kadomtsev-Petviashvili equations which depend explicitly on x and t. However, different from [25], starting with the Bäcklund transformation [(14) and (15)], we will derive and investigate the inverse scattering scheme [2, 23, 24] for (10) with the aid of symbolic computation. Computer algebra for this type of calculation can be referred to [22].

Setting $g = \psi f$ with $u(x,y,t) = (D_x^2 f \cdot f)/f^2 = 2(\mathrm{Log} f)_{xx}$ in (14) and (15) leads to

$$6\psi_{xx} + 6u\psi - 6B(t)\psi_y - a(t)x\psi - \lambda(t)\psi = 0, \quad (16)$$

$$2\psi_t + 2\psi_{xxx} + 6u\psi_x + 6B(t)\psi_{xy} + 6B(t)\psi\int u_y \mathrm{d}x + a(t)x\psi_x + \lambda(t)\psi_x - \mu(y,t)\psi = 0, \quad (17)$$

which constitute the inverse scattering scheme for (10).

By introducing two linear differential operators, $\widehat{L}_1$, $\widehat{L}_2$, defined by

$$\widehat{L}_1 = \partial_y - \frac{1}{B(t)}\partial_{xx} - \frac{u}{B(t)} + \frac{\lambda(t)}{6B(t)} + \frac{a(t)x}{6B(t)},$$

$$\widehat{L}_2 = \partial_t + \partial_{xxx} + 3u\partial_x + 3B(t)\partial_{xy}^2 + 3B(t)\partial_x^{-1}u_y + \frac{a(t)x}{2}\partial_x + \frac{\lambda(t)}{2}\partial_x - \frac{\mu(y,t)}{2},$$

(16) and (17) may be expressed as $\widehat{L}_1\psi = 0$ with $\widehat{L}_2\psi = 0$. In other literature [2, 23, 24], the pair of the linear differential operators $(\widehat{L}_1, \widehat{L}_2)$ is called Lax pair, which satisfies a compatibility condition if their order may be interchanged as $\widehat{L}_1\widehat{L}_2\psi = \widehat{L}_2\widehat{L}_1\psi$ or, equivalently, $[\widehat{L}_1, \widehat{L}_2] = \widehat{L}_1\widehat{L}_2 - \widehat{L}_2\widehat{L}_1 = 0$.

Here, the commutator of the operators $\widehat{L}_1$ and $\widehat{L}_2$ gives the generalized variable-coefficient two-dimensional KdV model with external-force term, (10), provided that the following two constraints on $a(t)$, $B(t)$ [or $b(t)$], $\lambda(t)$, $\mu(y,t)$ and $F_{xx}(x,y,t)$ are satisfied:

(1) $$\frac{-2a(t)}{B(t)} - \frac{B'(t)}{B^2(t)} = 0 \Leftrightarrow \frac{B'(t)}{B(t)} = -2a(t),$$
i. e., $b(t) = 3\alpha \mathrm{e}^{-4\int a(t)\mathrm{d}t}$,

where α is an arbitrary integration constant;

(2) $$-\frac{a^2(t)x}{3} - \frac{a'(t)x}{6} - \frac{a(t)\lambda(t)}{3} - \frac{\lambda'(t)}{6} - \frac{\mu_y(y,t)B(t)}{2} = F_{xx}(x,y,t).$$

Constraint (1) determines the relationship between the variable-coefficient functions $a(t)$ and $b(t)$, while (2) constrains the functional forms of the variable-coefficient functions and the external-force term. Under these two constraints, we may claim that (10) is integrable in the sense of having an inverse scattering scheme [(16) and (17)] or a Lax pair.

As shown previously, the inverse scattering scheme and the bilinear method have a strong relation from the viewpoint of the Bäcklund transformation. *It deserves to be specially noted that the bilinear Bäcklund transformation not only gives rise to the original equation as its compatibility condition, but also generates certain constraints on the variable-coefficient functions and the external term, in virtue of the inverse scattering scheme.*

4. Applications and Examples

We have extended the bilinear method to (10) and derived its bilinear Bäcklund transformation and inverse scattering scheme with certain corresponding constraints. Attention should be paid to these constraints for further study on (10). Combining with constraint (1), let us concentrate on constraint (2) to make some discussion and give several examples of physical interest in this section. Noting that $3B^2(t) = b(t)$, for convenience, we conduct our work with respect to $B(t)$ [instead of $b(t)$].

(i) $F_{xx}(x,y,t) = 0$.

In this case, combining constraint (1) with (2), we calculate: $a(t) = \frac{1}{2t+\beta}$, $b(t) = \frac{3\alpha}{(2t+\beta)^2}$, $\mu(y,t) = y\mu_1(t) + \mu_2(t)$, and $\lambda(t) = \frac{-3}{2t+\beta}\{\int[\mu_1(t)B(t)(2t+\beta) + \gamma]\mathrm{d}t\}$, where β and γ are two arbitrary integration constants, while $\mu_1(t)$ and $\mu_2(t)$ are two arbitrary functions of the variable t.

Thus, (10) reduces to (9), provided that $\alpha = 4\sigma^2$ with $\beta = 0$, and turns to (5) by setting $\alpha = 0$ with $\beta = 0$. This means that some completely integrable models, such as the cylindrical Kadomtsev-Petviashvili equation and the cylindrical KdV equation, can be included in our model under the aforementioned two constraints. Furthermore, (14) with (15) and (16) with (17) [after corresponding reduction] can be, respectively, regarded as the Bäcklund transformation and inverse scattering scheme for these equations.

(ii) $F_{xx}(x,y,t) \neq 0.$

Case 1. $-\dfrac{a^2(t)x}{3} - \dfrac{a'(t)x}{6} \neq 0 \quad \Rightarrow \quad a(t) \neq \dfrac{1}{2t+\beta}.$

In virtue of constraint (2), we can deduce that the external-force term, $F_{xx}(x,y,t)$, should obey the formula

$$F_{xx}(x,y,t) = xF_1(t) + F_2(y,t) + F_3(t),$$

where $F_1(t) = -\frac{a^2(t)}{3} - \frac{a'(t)}{6}$, $F_2(y,t) = -\frac{\mu_y(y,t)B(t)}{2}$ and $F_3(t) = -\frac{a(t)\lambda(t)}{3} - \frac{\lambda'(t)}{6}$. Hereby, (10) reduces to

$$\begin{aligned}&[u_t + 6uu_x + u_{xxx}]_x + a(t)u_x + b(t)u_{yy} + xF_1(t)\\ &+ F_2(y,t) + F_3(t) = 0.\end{aligned} \tag{18}$$

(Note that the external-force term here depends on the variables x, y and t.)

A special instance for this case has the form

$$\begin{aligned}&[u_t + 6uu_x + u_{xxx}]_x + a(t)u_x + b(t)u_{yy}\\ &+ xF_1(t) + F_3(t) + F_4(t) = 0\end{aligned} \tag{19}$$

with $F_4(t) = -\frac{1}{2}\mu_1(t)B(t)$, which means that $F_{xx}(x,y,t)$ is independent of the variable y. (Note that the external-force term here depends on the variables x and t.)

Case 2. $-\dfrac{a^2(t)x}{3} - \dfrac{a'(t)x}{6} = 0 \quad \Rightarrow \quad a(t) = \dfrac{1}{2t+\beta}.$

In this case, starting with constraint (2), we can infer that the external-force term, $F_{xx}(x,y,t)$, should possess the form [$F_2(y,t)$ and $F_3(t)$ defined here are similar to these in case 1]

$$F_{xx}(x,y,t) = F_2(y,t) + F_3(t)$$

with a subsidiary condition: $F_2(y,t)F_3(t) \neq 0$, which demands that:

(1) $\lambda(t) = \delta e^{-2\int a(t)dt}$ with $\mu_y(y,t) \neq 0$ or
(2) $\lambda(t) \neq \delta e^{-2\int a(t)dt}$ with $\mu_y(y,t) = 0$ or
(3) $\lambda(t) \neq \delta e^{-2\int a(t)dt}$ with $\mu_y(y,t) \neq 0$,

where δ is an arbitrary integration constant.

Hence, (10) can be expressed as

$$\begin{aligned}&[u_t + 6uu_x + u_{xxx}]_x + a(t)u_x + b(t)u_{yy}\\ &+ F_2(y,t) + F_3(t) = 0.\end{aligned} \tag{20}$$

(Note that the external-force term here depends on the variables y and t.)

Two special examples for this case are listed as follows (note that the external-force term in each example only depends on the variable t):

- $\lambda(t) = \delta e^{-2\int a(t)dt}$ with $\mu(y,t) = y\mu_3(t) + \mu_4(t)$, where $\mu_3(t) \neq 0$ and $\mu_4(t)$ are two arbitrary functions of t; then $F_{xx}(x,y,t)$ is of the form as $F_{xx}(x,y,t) = -\frac{1}{2}\mu_3(t)B(t) \equiv F_5(t)$ and (10) reduces to

$$[u_t + 6uu_x + u_{xxx}]_x + a(t)u_x + b(t)u_{yy} + F_5(t) = 0. \tag{21}$$

- $\lambda(t) \neq \delta e^{-2\int a(t)dt}$ with $\mu(y,t) = y\mu_5(t) + \mu_6(t)$, where $\mu_5(t)$ and $\mu_6(t)$ are two arbitrary functions of t; then $F_{xx}(x,y,t)$ is of the form as $F_{xx}(x,y,t) = -\frac{a(t)\lambda(t)}{3} - \frac{\lambda'(t)}{6} - \frac{\mu_5(t)B(t)}{2} \equiv F_6(t)$ and (10) becomes

$$[u_t + 6uu_x + u_{xxx}]_x + a(t)u_x + b(t)u_{yy} + F_6(t) = 0. \tag{22}$$

It has to be noticed that all the previous investigations have been handled on the basis of constraints (1) and (2). As a result, various cases of the external-force term are deduced in detail. As shown in Section 3, the two constraints are closely related to the inverse scattering scheme or Lax pair, that is, (10) is thought to be integrable under these constraints. Hence, we end up with a conclusion that all the cases obtained above are integrable (in the sense of having an inverse scattering scheme or a Lax pair).

5. Conclusions and Discussion

In conclusion, we have investigated a generalized variable-coefficient two-dimensional KdV model with external-force term, i. e., (10). With the aid of symbolic computation, we have extended the bilinear method to this model and obtained the bilinear form and Bäcklund transformation, through which the inverse scattering scheme (Lax pair) with corresponding constraints has been derived. Attention should be emphasized on the following:

- The generalized variable-coefficient two-dimensional KdV model with external-force term, (10), has abundant applications in various branches of physical and engineering sciences as mentioned before. Many important models appearing in shallow water waves, plasma physics, and fluid dynamics, for example *the cylindrical KdV equation* and *the cylindrical Kadomtsev-Petviashvili equation*, can be included by (10). Therefore, the investigations upon (10) are of great interest and significance.

• The bilinear method employed in the present paper is a straightforward and effective one, which was originally proposed by R. Hirota and applied to constant-coefficient NLEEs, while here we have developed it to a variable-coefficient two-dimensional model. It means that the bilinear method is also powerful in dealing with some *variable-coefficient* and *higher-dimensional* problems, and studies on this point should be furthered.

• The integrability of (10) turns to be a significant matter. For many completely integrable NLEEs, the extension of their coefficients from constants to functions often makes them non-integrable. Hence, certain constraints on the variable-coefficient functions are needed to be constructed in hope that the variable-coefficient NLEEs become still integrable. In the present paper, based on the Bäcklund transformation and the inverse scattering scheme, two constraints on the variable-coefficient functions of (10) have been given and, via symbolic computation analysis of the constraints, various cases thought to be integrable have been listed. *These cases with suitable choices of the variable-coefficient functions for each specific problem could be used in many more complicated, realistic physical situations and experimental environments, considering the inhomogeneities of media, nonuniformities of boundaries and external forces. For example, many Kadomtsev-Petviashvili equations with external-force term (only depending on the variable t) can be employed to model soliton dynamics in media with low-frequency stochastic fluctuations, where the external-force term describes the external noise when the characteristic lengths of the soliton are much smaller than the coherent length of noise (for details see [13]).*

From the above arguments, we hope that the Bäcklund transformation, the inverse scattering scheme with associated constraints, and the examples of the model obtained in this paper will be of certain value for further studies on the variable-coefficient two-dimensional KdV model with various external-force terms.

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